Splitting in the Category of Groups

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Group is a bad category. It has kernels, cokernels, free objects, projective objects, injective objects, short exact sequences, and even splitting, but these things are far from nice. **Group** is like that one kid that hangs around the cool kids (the Abelian categories) and hopes that some of their coolness will rub off, but instead it just looks like more of a loser than it actually is.

1 Short Exact Sequences

So what is a short exact sequence? It's not that different from the situation in any module category you like:

$$1 \to K \to G \to H \to 1$$

with kernel equal to image at each spot, noting that we use 1 here as the trivial group. We have some immediate short exact sequences:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

$$1 \longrightarrow \mathbb{Z}_3 \longrightarrow S_3 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Q_8 \longrightarrow \mathbb{Z}_4 \longrightarrow 1$$

where the first maps are obvious, the maps in the second coming from the injection $A_3 \trianglelefteq S_3$, and the the third from $Z(Q_8) \trianglelefteq Q_8$.

In fact, every short exact sequence is of the form

$$1 \to N \to G \to G/N \to 1$$

We say that an **extension**[1] of a group H by a group K is a group G along with maps $i : K \to G, \pi : G \to H$ such that

$$1 \to K \xrightarrow{i} G \xrightarrow{\pi} H \to 1$$

is exact.
It's well-known that extensions in a module category are classified by the elements of first Ext, but no such result has appeared for groups. It's certainly clear that $K \times H$ is such a group, so there are always extensions.

2 Splitting

Much as we can talk about splitting sequences of modules, we can talk about splitting sequences of groups. Just as in the module theory, the natural place to start is with a section.

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, "i"] & G \arrow[r, "\pi"'] & H \arrow[l, bend right, "r"'] \arrow[r] & 1 \end{tikzcd}$$

with $\pi \circ r = id_H$.

In a module category, this is enough to say that $G \cong K \oplus H$. We wash our hands and we're done with it. The interesting example presents itself in the category of groups, though:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}_2 \arrow[r] & D_4 \arrow[r, "\pi"'] & \mathbb{Z}_4 \arrow[l, bend right, "r"'] \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \times \mathbb{Z}_4 \arrow[r, "\pi"] & \mathbb{Z}_4 \arrow[l, bend right, "r"'] \arrow[r] & 1 \end{tikzcd}$$

In the first example, we're modding out the reflection group. In the second, all the maps are obvious. Even though both groups split the sequence, they're not isomorphic. So in fact we can say that there are multiple non-isomorphic splits of two groups.

The situation becomes much more clear once we have the following structure theorem:
If a short exact sequence of groups splits on the right, then it is isomorphic to the same sequence containing a semidirect product: that is, the following diagram commutes

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, "i"] \arrow[d, "\mathrm{id}"'] & G \arrow[r, "\pi"'] \arrow[d, "\theta"'] & H \arrow[l, bend right, "r"'] \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & K \arrow[r] & K \rtimes_\varphi H \arrow[r, "\pi"'] & H \arrow[r] & 1 \end{tikzcd}$$

with the entire diagram commuting.
The action $\varphi : H \to \mathrm{Aut}K$ is defined by $i(\varphi_h(k)) = r(h)i(k)r(h^{-1})$, a well-defined action because the kernel is normal and i is an injection. Standard arguments about conjugation show that this is a homomorphism into the automorphism group; in fact, what we see here is an example of an **internal** semidirect product.
Now for the map θ, it's best to first see that $G = KH$, the internal product of $i(K)$ and $r(H)$. Take $g \in G$. $\pi \circ r \circ \pi(g) = \pi(g)$, so $g(r\pi g)^{-1} \in \mathrm{Ker}\,\pi = \mathrm{Im}\ i$. So $g(r\pi g)^{-1} = i(k) \implies g = i(k)r(\pi(g))$.
Now map $g \mapsto (k, \pi(g))$. (Note that the choice of k is unique by injectivity.) This is a group

homomorphism:

$$\begin{aligned}
\theta(g_1)\theta(g_2) &= (k_1, \pi(g_1))(k_2, \pi(g_2)) \\
&= (k_1\varphi_{\pi(g_1)}(k_2), \pi(g_1)\pi(g_2)) \\
&= (k_1 i^{-1}(r(\pi(g_1))i(k_2)r(\pi(g_1))^{-1}), \pi(g_1)\pi(g_2)) \\
&= (k_1 i^{-1}(r(\pi(g_1))i(k_2)r(\pi(g_1))^{-1}), \pi(g_1 g_2)) \\
&= \theta(g_1 g_2)
\end{aligned}$$

This last, surprising statement, is because the i of the first component is

$$i(k_1)r(\pi(g_1))i(k_2)r(\pi(g_1))^{-1}$$

This is an element of $i(K)$ with the property that

$$\begin{aligned}
i(k_1)r(\pi(g_1))i(k_2)r(\pi(g_1))^{-1}r(\pi(g_1))r(\pi(g_2)) &= i(k_1)r(\pi(g_1))i(k_2)r(\pi(g_2)) \\
&= g_1 g_2
\end{aligned}$$

But this decomposition was unique, so $i(k_1)r(\pi(g_1))i(k_2)r(\pi(g_1))^{-1}$ is the K-component of g_1g_2. So this is a homomorphism.
Now that we've put in all the work making sure that this thing actually does what we want it to do, it's trivial to see that it is an isomorphism. It's injective: if $g \mapsto (1,1)$, then $\pi(g) = 1$, so $g = i(1)r(\pi(1)) = 1$. It's also surjective: for some element $(k,h) \in K \rtimes H$, $g = i(k)r(h)$ has exactly the properties that we want.
Checking that this diagram commutes is diagram chasing, and I wouldn't deprive you of that fun. It's covered in more detail in the article from which I got most of this section, Keith Conrad's "Splitting of a Short Exact Sequence of Groups"[2].

The other part of an argument like this is "what happens if the sequence splits the other way"? The answer is actually the direct product. If we have a section $r : G \to K$, defining $\theta(g) = (r(g), \pi(g))$, θ is an isomorphism and the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "i"'] \arrow[d, "\mathrm{id}"'] & G \arrow[r, "\pi"'] \arrow[l, bend right, "r"'] \arrow[d, "\theta"'] & H \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\
1 \arrow[r] & K \arrow[r] & K \times H \arrow[r] & H \arrow[r] & 1
\end{tikzcd}$$

Again, the proof is standard. θ is an isomorphism by the universal property of the product. Clearly it is injective, and it is surjective: $i(k)\pi^{-1}(h) \in G$ for any $(k,h) \in K \times H$. The diagram again trivially commutes.
For those of you seeking intuition, you can think that r being this map, it must have kernel H. This means that H and K are both normal, and G is their internal product. Also, if you believe that there are more semidirect products than direct products (and there certainly aren't less), then you believe that there are less "left sections" than "right sections".

3 Projectives, Injectives, and Frees

Free groups are characterized by the following universal property: If S is a set, G a group, then $\exists! \phi : F_S \to G$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & G \\ F_S \arrow[ur, dashed, "\phi"] & \end{tikzcd}$$

Just as in the case of modules, the fact that we can define a map anywhere we want means that free modules always split sequences: if we have

$$1 \to K \to G \xrightarrow{\phi} F_S \to 1$$

then we can just pick $g_s \in \phi^{-1}(s)$ and define $r(s) = g_s$ and extend this to a section via the universal property. Note that although we have picked the morphism, any section gives rise to an isomorphic semidirect product.

Any experience with homological algebra will grant exposure to the concept of a **projective** object. This is a group P with the following (not universal) property: for any groups G, H, with a surjective homomorphism $\pi : G \to H$ and a map $\phi : P \to H$, there is some (not necessarily unique) map $\theta : P \to G$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\theta"'] \arrow[d, "\phi"] \\ G \arrow[r, two heads, "\pi"] & H \end{tikzcd}$$

Notably, if we have a short exact sequence

$$1 \to K \to G \xrightarrow{\phi} P \to 1$$

taking $H = P, \psi = \mathrm{id}$ in the diagram will give a section and therefore a split. So why don't we talk about projective groups? Because of this short exact sequence:

$$1 \to K \to F_I \to P \to 1$$

where I is the indexing set formed by elements of P. Simply put, any free group surjects onto any group, and if the target space is projective, the injective section $r : P \to F_I$ makes P a subgroup of a free group, and therefore free. So every projective object in **Group** is free.

The situation worsens for "injective" groups. An injective group Q in **Group** is an object with the following (not universal) property: for any injection $i : K \to G$, $\phi : K \to Q$, there is a (not necessarily unique) map $\theta : G \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, hook, "i"] \arrow[d, "\phi"] & G \arrow[dl, dashed, "\theta"] \\ Q & \end{tikzcd}$$

We can actually rule out injectives very easily using our statements from above. First, note that every injective must split every sequence that it's the first component of, taking $K = Q$, $\phi = \mathrm{id}$. Now consider $G = Q \rtimes \mathrm{Aut}\, Q$, which has the associated short exact sequence:

$$1 \to Q \to Q \rtimes_{\mathrm{id}} \mathrm{Aut}\, Q \to \mathrm{Aut}\, Q \to 1$$

The sequence splits, so we then have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & Q \arrow[r] \arrow[d] & Q \rtimes \mathrm{Aut}\, Q \arrow[r] \arrow[d] & \mathrm{Aut}\, Q \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Q \arrow[r] & Q \times \mathrm{Aut}\, Q \arrow[r] & \mathrm{Aut}\, Q \arrow[r] & 1
\end{tikzcd}$$

where all the maps are natural. But this means that the faithful action of $\mathrm{Aut}\, Q$ on Q is trivial, so $\mathrm{Aut}\, Q = 1$. A bit of basic group theory gives that this means that $Q \cong \mathbb{Z}_2$ or $Q = 1$, and $\mathbb{Z}_2$ isn't injective in the category of Abelian groups, a full subcategory, so there's no way that it's injective in the category of groups. So the only injective is 1.

4 Algebraic Topology

It turns out that just as with homological algebra, we can apply these statements to topological spaces. (Joke about homotopical algebra.) One interesting application is to calculating fundamental groups. Consider the following two spaces (draw the standard pictures) of T^2 and K, the torus and the Klein bottle. We actually have a continuous map from both of them to S^1, mapping into a medial circle. This induces the following short exact sequence of groups for both of them:

$$1 \to F_1 \to \pi_1(T), \pi_1(K) \to F_1 \to 1$$

and, as these sequences split, we know that $\pi_1(T), \pi_1(K) \cong F_1 \rtimes_\psi F_1$. These aren't the same group, because the actions of the right copy of F_1 are different: $\mathrm{Aut}\, F_1 \cong \mathbb{Z}_2$. The Klein bottle has the nontrivial action, and the torus the trivial one.

We can also mine some information from algebraic topology. The proof that subgroups of free groups is generally proved topologically, but there's one nice example which proves that rank isn't an additive invariant. Consider the kernel of the abelianization map $\phi : F_{\{a,b\}} \to \mathbb{Z}^2$. $\mathrm{Ker}\, \phi$ is a subgroup of $\pi_1(S^1 \vee S^1)$, so there's a covering space associated to it. A homotopy of this space – crushing a maximal tree – gives that it's homotopic to $\vee_4 S^1$, with fundamental group F_4.

5 Back to Extensions

Splitting isn't enough to characterize extensions. Q_8 is an excellent example: its only normal subgroup is its center, but it sits in this short exact sequence:

$$1 \to \mathbb{Z}_2 \to Q_8 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

The fact that there's no splitting is evident from the fact that Aut $\mathbb{Z}_2 \cong 1$, so if the sequence did split it would be Abelian, but Q_8 is non-Abelian.

References

[1] "nLab group extension." Group extension in nLab, `ncatlab.org/nlab/show/group extension`.

[2] Conrad, Keith. "Splitting of Short Exact Sequences for Groups." Keith Conrad. Accessed February 23, 2018. `http://www.math.uconn.edu/ kconrad/blurbs/grouptheory/splittinggp.pdf`